) What is a group?

Ans: A group is a set of distinct elements, $G \equiv \{ E, A, B, C, D, \ldots \}$, endowed with a law of composition such as addition, multiplication, matrix multiplication etc.

The number of elements in a group is called its order.

Some examples of group are

i) The group of order two consisting of the real numbers 1, -1 with ordinary multiplication as the law of composition.

ii) The group of order four consisting of the complex numbers 1, i, -1, -i (where $i^2 = -1$) under multiplication.

The properties of Group

a) The composition of any two elements A and B of G under the given law results in an element which also belongs to G. Thus

$$A \cdot B \in G, \quad B \cdot A \in G \longrightarrow (1)$$

where we have denoted the . composition of two elements of G by the symbol $\cdot$

Symbolically, $A \cdot B \in G \; \forall \; A, B \in G \longrightarrow (2)$

This property is known as the closure property

of the group and the set is said to be closed under the given law of composition.

b) There exists an identity element $E \in G$ such that for all $A \in G$,

$$E \cdot A = A \cdot E = A$$

Symbolically

$$\exists\, E \in G \ni E \cdot A = A \cdot E = A \quad \forall A \in G$$

E is known as the identity element of G.

c) For any element $A \in G$, there exists a unique element $B \in G$ such that

$$A \cdot B = B \cdot A = E$$

symbolically,

$$\forall A \in G \;\exists\, B \in G \ni A \cdot B = B \cdot A = E$$

B is called the inverse of A and vice versa

d) The law of composition of the group elements is associative, i.e. for any $A, B, C \in G$.

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

symbolically, $A \cdot (B \cdot C) = (A \cdot B) \cdot C \quad \forall A, B, C \in G$

Other examples of group

a) The discrete infinite group of all real integers. The law of composition is addition and the identity element is 0.

b) The set of all real numbers under addition. This is a continuous group with 0 as the identity element. The inverse of a number b is its negative $-b$.

c) The set of all positive (zero excluded) real numbers under multiplication. The identity element is 1 and the inverse of x is its reciprocal $1/x$

d) The single point set containing just the unity is a group of order one under multiplication.

e) The set of two matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ under matrix multiplication.

f) The set of all nonsingular square matrices of order n (n a positive integer) under matrix multiplication.

g) If K is a positive integer, the set $(0, 1, 2, \cdots, K-1)$ of K integers is a group under addition modulo (K).

The identity element is zero and the inverse of an element r is $k-r$.

h) The set of all matrices of order $m \times n$ under matrix addition. The identity element is the null matrix of order $m \times n$ and the inverse of an element A is its negative $-A$.

Subgroup :-

A set H is said to be a subgroup of a group G if H is itself a group under the same law of composition as that of G and if all the elements of H are also in G.

As an example, Consider the four elements (E, C_4, C_4^2, C_4^3) of C_{4v}. It is easy to see that this set satisfies all the axioms defining a group, hence it is a subgroup of C_{4v}. Some more examples of the subgroups of C_{4v} are (E, C_4^2, m_x, m_y), (E, σ_u) etc.

Cyclic groups :-

If A is an element of a group G, all integral powers of A such as $A^2, A^3, \ldots$, must also be in G. If G is a finite group there must exist a finite positive integer n such that $A^n = E$, the identity element.

The group $(A, A^2, A^3, \ldots, A^n \equiv E)$ has the property that each of its elements is some power of one particular element. Such groups are called cyclic groups.

That is a group generated by a single element is a cyclic group.

Cosets :-

Q. Obtain the direct products of all the irreducible representation of the group C_{3v} and reduce them into direct sums of the irreducible representation.

Products of irreducible representations may give reducible representations.

Direct products are expressible as direct sums of irreducible representations

Examples in C_{3v}

The character table of C_{3v} is as follows

C_{3v}	E	$2C_3$	$3\sigma_v$
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0

On determining the representation of a multi-electron wave function gives a means to decompose a multielectron wave function into terms thus gives the character table as

C_{3v}	E	$2C_3$	$3\sigma_v$
$E \otimes E$	4	1	0
$A_1 \oplus A_2 \oplus E$	4	1	0

That is $E \otimes E = A_1 \oplus A_2 \oplus E$

Again inner product of the projection operator with the reducible representation gives the number of representations present in the reducible representation e.g in C_{3v}

$$P_{A_1} \cdot (E \otimes E) = \frac{1}{6}(1,2,3) \cdot (4,1,0)$$

$$= \frac{1}{6}(1\cdot 4 + 2\cdot 1 + 3\cdot 0)$$

$$= \frac{6}{6} = 1$$

$$P_{A_2} \cdot (E \otimes E) = \frac{1}{6}(1,2,-3) \cdot (4,1,0)$$

$$= \frac{1}{6}(1\cdot 4 + 2\cdot 1 - 3\cdot 0)$$

$$P_E \cdot (E \otimes E) = \frac{6}{6} = 1$$

$$= \frac{1}{6}(2,-2,0) \cdot (4,1,0)$$

$$= \frac{1}{6}(2\cdot 4 + -2\cdot 1 + 0\cdot 0)$$

$$= \frac{6}{6} = 1$$

[projection operator is formed for an irreducible representation by multiplying the number of operations in the class times the character

$$P_{A_1} = \frac{1}{6}(1\times 1, 2\times 1, 3\times 1) = \frac{1}{6}(1,2,3)$$

$$P_{A_2} = \frac{1}{6}(1\times 1, 2\times 1, 3\times -1) = \frac{1}{6}(1,2,-3)$$

$$P_E = \frac{1}{6}(1\times 2, 2\times -1, 3\times 0) = \frac{1}{6}(2,-2,0)$$]

Product of two degenerate irreducible representation is a reducible representation i.e $E \otimes E = A_1 \oplus A_2 \oplus E$. Thus products of reduced irreducible representation is as direct sum of irreducible representation.

- Other direct products i.e $E \times A_1$, $E \times A_2$, $A_1 \times A_2$ are not reducible.

Since $A_1 \times A_1 = A_2 \times A_2 = A_1$ and $A_1 \times A_2 = A_2$, we have

$$\left(\begin{array}{cc|c} A_1 & A_1 & A_1 \\ 1 & 1 & 1 \end{array}\right) = \left(\begin{array}{cc|c} A_1 & A_2 & A_2 \\ 1 & 1 & 1 \end{array}\right) = \left(\begin{array}{cc|c} A_2 & A_2 & A_1 \\ 1 & 1 & 1 \end{array}\right) = 1$$

Recall $A_1 \times E = A_2 \times E = E$, we then have

$$\left(\begin{array}{cc|c} A_1 & E & E \\ 1 & \nu & \xi \end{array}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\nu\xi}, \quad \left(\begin{array}{cc|c} A_2 & E & E \\ 1 & \nu & \xi \end{array}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\nu\xi}$$

Finally, $E \otimes E = A_1 \oplus A_2 \oplus E$ and we have

$$\left(\begin{array}{cc|c} E & E & A_1 \\ \mu & \nu & 1 \end{array}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\mu\nu}, \quad \left(\begin{array}{cc|c} E & E & A_2 \\ \mu & \nu & 1 \end{array}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\mu\nu}$$

and

$$\left(\begin{array}{cc|c} E & E & E \\ \mu & \nu & 1 \end{array}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mu\nu}$$

$$\left(\begin{array}{cc|c} E & E & E \\ \mu & \upsilon & 2 \end{array}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}_{\mu\nu}$$

Characteristics of cyclic groups

1) Every cyclic group is abelian

Proof: Let a be a generator of a cyclic group G and let $a^r, a^s \in G$ for any $r, s \in I$

then $a^r \cdot a^s = a^{r+s} = a^{s+r} = a^s \cdot a^r$

Because $r+s = s+r$ for $r, s \in I$

Thus the operation is commutative and hence the cyclic group G is abelian.

2) The order of a cyclic group is the same as the order of its generator.

Proof: Let the order of a generator a of a cyclic group be n, then

$$a^n = e \quad \text{while } a^s \neq e$$

for $0 < s < r$.

When $s > n$, $s = nq + r$, $0 \leq r < n$ (say),

We observe that

$$a^s = a^{nq+r} = (a^n)^q \cdot a^r = a^q \cdot a^r$$
$$= e \cdot a^r = a^r$$

Thus there are exactly n elements in the group by a^r, where $0 \leq r < n$. Therefore there are ~~n and~~ only n distinct elements in the cyclic group i.e the order of group is n.

3) The generators of a cyclic group of order n are all the elements a^p, p being prime to n and $0 < p < n$

Proof :- We know that $(a^p)^n = (a^n)^p = e$

$\therefore$ The order of a^p is n.

Also $(ap)^s = ap^s \neq e$ if $0 < p < n$ because n does not divide p, nor does it divide s, therefore it does not divide ps

Now let,

$$ps = nq + r, \quad 0 < r < n$$

and $(a^p)^s = a^{ps} = a^{nq+r}$

$$= (a^n)^q \cdot a^r = e^q a^r$$

$$= e \cdot a^r = a^r \neq 0$$

Thus, a^{in} is a generator of the group.

* Under what condition a group will be a cyclic group ?

→ A group which has the property that each of its elements is some power of one particular element is called cyclic group. And cyclic groups are abelian while the converse is not necessarily true.

Show that C_{4v} is homomorphic to the group (1, -1) under multiplication. Also show that this 4 to 1 homomorphic mapping can be established in 3 distinct ways.

→ The group C_{4v} has a four to one correspondence with the ordinary multiplicative group containing only 1 and -1 :

$E, C_4^2, m_x, m_y \leftrightarrow 1$ and $E, C_4^2, m_x, m_y \leftrightarrow -1$

i.e Group (E, C_4^2, m_x, m_y) Group $(1, -1)$

$E, C_4^2, m_x, m_y \longrightarrow 1$

$E, C_4^2, m_x, m_y \longrightarrow -1$

4 to 1 homomorphic mapping

We may get it in three distinct ways as follows.

1) $E \leftrightarrow 1$, $C_4^2 \leftrightarrow 1$, $m_x \leftrightarrow -1$, $m_y \leftrightarrow -1$

2) $E \leftrightarrow 1$, $C_4^2 \leftrightarrow -1$, $m_x \leftrightarrow -1$, $m_y \leftrightarrow 1$

3) $E \leftrightarrow 1$, $C_4^2 \leftrightarrow -1$, $m_x \leftrightarrow 1$, $m_y \leftrightarrow -1$

This is homomorphism, possessed by all groups, is obtained when every element is assigned to correspond to the identity.

Show that every subgroup of index 2 is a normal subgroup.

→ Let H be a subgroup of index 2.
Then the group G acts on the left cosets G/H by left multiplication.

It induces the permutation representation

$$\rho : G \to S_2$$

Let $K = \ker \rho$ be the kernal of ρ.

Since $kH = H$ for $k \in K$, we have $K \subset H$.

Let $[H : K] = m$

By the first isomorphism theorem, the quotient group G/K is isomorphic to the subgroup of S_2, thus $[G : K]$ divides

$|S_2| = 2!$ by Lagrange's theorem.

Since $[G:K] = [G:H][H:K] = 2m$, we have

$2m \mid 2!$ and hence $m \mid (2-1)!$

If m has a prime factor q, then $q \geq 2$ since the factors of $(2-1)!$ are only prime numbers less than 2.

Thus $m \mid (2-1)!$ implies that $m = 1$, hence $H \subset K$ Therefore H is normal since a kernal is always normal.

Mapping

Defⁿ

Assign to each even integer the value 1, and to each odd integer the value −1. Let us give the name α to this assignment; thus α assigns to each even element in Z, the set of all integers, the unique element +1 in the set $\{1, -1\}$ and to each odd element in Z the unique element −1 in $\{1, -1\}$. In less detailed terms α assigns to each element in Z a unique element in $\{1, -1\}$. Such an assignment is termed a mapping from Z into $\{1, -1\}$ or a map from Z into $\{1, -1\}$. More generally, if S and T are any two non-empty sets, a mapping or a map from S into T is an assignment of a unique element of T to each element of S. For the most part we shall denote mappings by lower case Greek letters such as α, β, γ. If α is a mapping from S into T, we shall express this fact more briefly by writing $\alpha : S \to T$; this is read "α is a mapping from S into T". We call S the domain and T the codomain of α.